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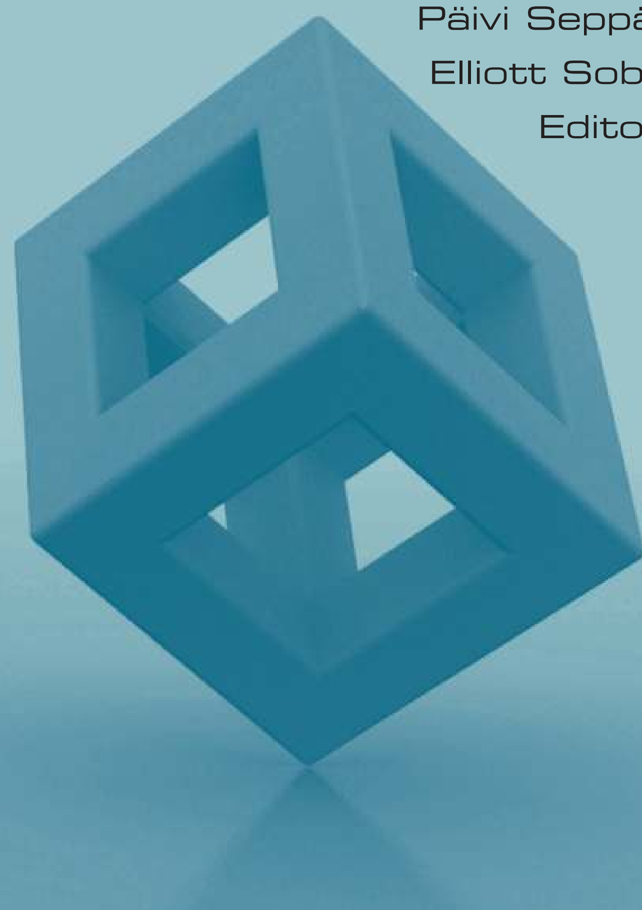

7 Intensional logic before Leibniz

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Abstract. Following on work by Klaus Glashoff, Leibniz's intensional reading of categorical propositions is interpreted within a formal framework in which intensions are related by inseparability and incompatibility, and some intensions are designated as quiddities. This same framework is used to demonstrate some of the main claims made in the modal logics of Avicenna and Robert Kilwardby. Those two logics are differentiated from each other by the different truth-conditions they assign to modal propositions, and by the different assumptions they make about quiddities. It is shown that on Leibniz's intensional reading, categorical propositions are equivalent to one of the main types of modal proposition distinguished by the medievals.

Keywords: Intensional logic, Leibniz, Avicenna, Kilwardby, syllogistic.

1 Introduction

Leibniz knew that a predicate's intension can be specified through the predicate's 'superconcepts', i.e. through the concepts that are inseparable from it as 'animal' is inseparable from 'man'.

Leibniz takes as his model of universal affirmative propositions a statement predicating a genus of its species. The genus is one factor in the definition of the species, the differentia being the other factor:

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> Thus as the notion of the genus is the part, the notion of the species is the whole, for it is composed from the genus and the differentia. (Glashoff, 2002, p.163. My translation.)

He thought that such propositions should be given an intensional reading:

> The common manner of statement concerns individuals, whereas Aristotle's refers rather to ideas or universals. For when I say *Every man is an animal* I mean that all the men are included amongst all the animals; but at the same time I mean that the idea of animal is included in the idea of man. 'Animal' comprises more individuals than 'man' does, but 'man' comprises more ideas or more attributes: one has more instances, the other more degrees of reality; one has the greater extension, the other the greater intension. (Leibniz, 1875–1890, p.495. Lenzen translation.)

According to this reading the truth of a categorical proposition depends on a relationship between the intension of the subject and that of the predicate. Klaus Glashoff illustrates this relationship by reference to the a-proposition, which Leibniz analyses as stating that everything 'positively contained' in the predicate is positively contained in the subject, and everything 'negatively contained' in the predicate is negatively contained in the subject. (Glashoff, 2010, p.267.)

Leibniz worked out a mathematical representation of these relationships. As Glashoff puts it:

> Leibniz's basic idea was to assign to each 'simple' concept a prime number and to each 'composed' concept the product of prime numbers. This idea leads directly to his definition of the U.A.-*propositio* – namely, that *Axy* is true just in case *y* divides *x*. (Glashoff, 2002, p.172.)

The truth-condition of the particular negative is the negation of that for the universal affirmative. But Leibniz had difficulties in formulating the truth-conditions for the universal negative and particular affirmative propositions. He first tried supposing that the particular affirmative *Some A is B* is true iff either *All A* is *B* or *All B* is *A* is true. In his arithmetical representation, what this comes to is that the number associated with *A* divides the number associated with *B* or vice versa. (Glashoff, 2002, p.173.) But he rejected this thought on the ground that a species might fall under two different genera neither of which was subalternate to the other.

Glashoff documents Leibniz's attempts to surmount this difficulty. In his final account of the truth-conditions for propositions with subject *x* and predicate *y*, Leibniz associated with each term an ordered pair of characteristic numbers, one marked as positive, the other negative. For example, $+s-\sigma$ might be associated with *x*, and $+p-\pi$ with *y*. As Glashoff explains it, the idea is that *s* is the product of prime numbers of elementary properties *contained* in *x*, that is, of all elementary properties *y* such that *All x* is *y* holds true. On the other hand, σ is the product of all primes belonging to those elementary properties *z* such that *No x is z* holds. Similarly for *p* and π. *s* and σ are stipulated to be 'apt', i.e. they cannot have a common divisor; they must be relatively prime. Similarly for p and π. Glashoff points out that these pairs correspond to the rational numbers. (Glashoff, 2002, p.177.) Using apt pairs of numbers, Leibniz gives the following truth-conditions for universal affirmative and particular affirmative propositions:

> Universal Affirmative. $\mathbf{a}((s,\sigma),(p,\pi))$ if and only if: p divides s and π divides σ.
> Particular Affirmative. $\mathbf{i}((s,\sigma),(p,\pi))$ if and only if: s and π as well as σ and p are relatively prime. (Glashoff, 2002, p.178.)

The i-proposition means that nothing positively contained in one term is negatively contained in the other (Glashoff, 2010, p.267.) Glashoff explains the conditions under which the truth-condition for the i-proposition follows from that for the a-proposition (Glashoff, 2002, p.183.)

My interest in this paper is not with the mathematics but with the underlying philosophical analysis in Leibniz's theory: his appeal to 'ideas' or 'concepts' as the intensions of predicates, his quantification over them (as in his phrase 'everything positively contained in the predicate'), and his claim that relations between these 'ideas' provide the truth-conditions for categorical predications read intensionally (e.g. the claim that the universal affirmative is true iff everything positively contained in the predicate is positively contained in the subject and everything negatively contained in the predicate is negatively contained in the subject).

2 A formal framework for a logic of intensions

With this is mind we need to devise a formal framework in which individual intensions can be named and quantified over. The framework needs to contain predicate constants which when attached to names of intensions are capable of producing truths. These predicate constants need to include some which will be useful for generating formal representations of (at least some) categorial propositions.

As a formal framework I shall suppose an extension of a predicate logic with identity, containing predicates $A, B, C, \ldots$. We augment this logic's object language in three ways. First, we add names corresponding to the predicates—α corresponding to A, β to B, γ to $C, \ldots$; these names are intended to name the intensions of their associated predicates; thus, α names the intension of the predicate A, and so on. Second, we add a one-place predicate Q which, when supplemented by the name of an intension, forms a proposition. The intended reading of Q is '_ is a quiddity'. In the Aristotelian philosophical tradition a quiddity is whatever is an appropriate answer to the philosophical question 'What is it? (*quid est*). Third, we add two-place predicates $\Leftarrow$, $\Downarrow$, from which propositions are formed by the addition of an ordered pair of names of intensions. The predicates are intended to express relations of inseparability and incompatibility between intensions. These two-place predicates are subject to the following postulates:

P1. $\kappa \Leftarrow \kappa$
P2. $(\kappa \Leftarrow \lambda \wedge \lambda \Leftarrow \mu) \supset \kappa \Leftarrow \mu$
P3. $\kappa \Downarrow \lambda \supset \lambda \Downarrow \kappa$
P4. $(\kappa \Downarrow \lambda \wedge \lambda \Leftarrow \mu) \supset \kappa \Downarrow \mu$.
P5. $\kappa \Leftarrow \lambda \supset \neg\kappa \Downarrow \lambda$

P5 effectively restricts consideration to intensions that are self-consistent. The other postulates reflect an intuitive understanding of inseparability and incompatibility among intensions.

Quiddities may be thought to be related in ways that are more restricted than the relations that govern intensions in general. For example, P6 may be thought to hold for quiddities, though it is not true in general that all pairs of intensions have to be related by either inseparability or incompatibility.

P6. $(Q\kappa \wedge Q\lambda) \supset (\kappa \Downarrow \lambda \vee \kappa \Leftarrow \lambda \vee \lambda \Leftarrow \kappa)$.

Even more specific than the notion of a quiddity is the notion of a *nature*:

D1. $N\kappa =_{def} Q\kappa \wedge \forall\lambda[(Q\lambda \wedge \kappa \Leftarrow \lambda) \supset \lambda \Leftarrow \kappa]$

A nature is a most specific quiddity.

Using D1 and P6, we can prove the theorem:

T1. $(N\kappa \wedge Q\lambda) \supset (\kappa \Downarrow \lambda \vee \lambda \Leftarrow \kappa)$

Proof. Suppose κ is a nature and λ a quiddity. Since by D1 natures are quiddities, we can apply P6 and infer that κ stands to λ in a relation of either inseparability or incompatibility. But since κ is a nature, by D1 if $\kappa \Leftarrow \lambda$ then $\lambda \Leftarrow \kappa$. So either κ and λ are incompatible or λ is inseparable from κ.

In order to test the satisfiability of a set **S** of propositions in the object language, we make the following additions to the usual requirements for a model. There is a domain **D** of intensions, and a subdomain **E** (**E** $\subset$ **D**) of quiddities. To each name of an intension in **S** is assigned a member of **D**. To each of the predicates $A, B, C, \ldots$ in **S** is assigned a subset of **D**. To each name of an intension κ such that the proposition '$Q\kappa$' is in **S**, is assigned a member of **E**, these being the only members of **E**. To $\Leftarrow$ is assigned a set r_1, to $\Downarrow$ a set r_2, where r_1 and r_2 are sets of ordered pairs of members of **D**.

Let us now look at two medieval logicians who, in different ways, appealed to intensional considerations in their treatment of modal logic. The individual analyses of modalised categorical propositions proposed by our medieval authors will be given a formal representation in our extended predicate logic. Both of the medieval logicians I will discuss proposed a distinctive way of theorising modalised categorical propositions, i.e. propositions of the following eight forms:

La: Every _ is necessarily ...
Le: No _ is possibly ...
Li: Some _ is necessarily ...

> Lo: Some _ is not possibly ...
> Ma: Every _ is possibly ...
> Me: No _ is necessarily ...
> Mi: Some _ is possibly ...
> Mo: Some _ is not necessarily ....

3 Avicenna (d. 1037)

In his late work *Pointers and Reminders* the great Persian logician Avicenna presents the following threefold division of predicates:

> Predicates may be essential, implicate accidental, and separable accidental. Let us begin by defining (*ta'rif*) the essential. [Some] predicates are . . . constitutive of their subjects. . . . by "constitutive" . . . [I mean] the predicate which the subject needs for the realisation of its quiddity, and which is intrinsic to its quiddity, a part of it. This is like being a figure for a triangle, or corporeality for man. . . . conceiving body as body [and] . . . triangle as triangle. (Street, forthcoming; Dunyā, 1971, p.151.)

Though he refers to predicates 'body' and 'triangle' here, it is clear that Avicenna is talking about these predicates in respect of their *intensions*. This is clear from his phrases 'body as body and 'triangle as triangle'. His idea is that one intension may be essential to another, or it may be an implicate of the other without being essential to it, or it may be separable from it.

He elaborates on the latter two cases:

> As for the non-constitutive implicate (and it is singled out by the term 'implicate' (*lazim*), even though the constitutive is also an implicate), it is that which is associated with (*yaṣḥabu*) the quiddity while not being a part of it, like [the sum of the angles of] a triangle being equal to the sum of two right angles. (Street, forthcoming; Dunyā, 1971, p.158.)

By an implicate of a given intension Avicenna appears to mean an intension that is inseparable from it; implicates of a given intension are contrasted with intensions that are not inseparable from it. He also uses a notion of a quiddity. I shall take his talk of what is intrinsic to a quiddity, or a part of the quiddity, to mean what is inseparable from the quiddity.

Avicenna divides propositions expressing a necessary connection between predicate and subject into three classes: those where the necessity is absolute, those where the necessity is relative to the essence of the subject, and those that are relative to a description under which the subject is presented:

> Necessity may be absolute (*'ala l-iṭlaq*), as in *God exists*. Or it may be connected (*mu'allaqa*) to a condition (*šarṭ*). The condition is either

> perpetual [relative] to the existence of the substance [of the subject] (*ḏat*), as in *Man is necessarily a rational body* ... or the duration (*dawam*) of the subject's being described with what is set down with it, as in *All mobile things are changing*. (Street, forthcoming; Dunyā, 1971, p.265.)

He gives an example of a substantial necessity-proposition:

> For example, some bodies by necessity are moving, that is, as long as the substance of those some [bodies] exists. (Street, forthcoming; Dunyā, 1971, p.292.)

He has in mind the heavenly bodies, part of whose essence is to be in motion.

Plausibly, we can capture Avicenna's idea of propositions expressing descriptional necessities in our extended predicate calculus by adopting the following formal representations for universal necessity- and possibility-propositions with subject A and predicate B, 'All A is necessarily B' (for example, 'Everything walking is necessarily moving'), 'No A is possibly B', 'All A is possibly B', 'No A is necessarily B':

La: $\beta \Leftarrow \alpha$
Le: $\beta \Downarrow \alpha$
Ma: $\forall\kappa[\alpha \Leftarrow \kappa \supset \neg\beta \Downarrow \kappa]$
Me: $\forall\kappa[\alpha \Leftarrow \kappa \supset \neg\beta \Leftarrow \kappa]$.

L stands for necessity, M for possibility. α stands for the intension of 'A', β for the intension of 'B'. Thus, the La-proposition is true iff the intension of the predicate is inseparable from the intension of the subject; the Me-proposition is true iff the intension of the predicate is separable from every intension from which the intension of the subject is inseparable. And so on for the others.

Particular L- and M-propositions can be accounted for as the contradictories of universal propositions of the dual modality.

Given these formal representations of descriptional necessity- and possibility-propositions, we can show that the Le-proposition is convertible, as is the Li-proposition, and we can demonstrate the validity or invalidity of syllogistic inferences constructed from descriptional propositions. I take as an example the syllogism Barbara LML ('Every A is possibly B, every B is necessarily C, so every A is necessarily C'), which is invalid.

T2. (i) *Le-conversion* is valid, (ii) *Li-conversion* is valid, (iii) *Barbara LML* is invalid – where all propositions are descriptional

What is required to be proved is that
(i) $\beta \Downarrow \alpha$ is deducible from $\alpha \Downarrow \beta$;
(ii) $\exists\kappa[\beta \Leftarrow \kappa \wedge \alpha \Leftarrow \kappa]$ is deducible from $\exists\kappa[\alpha \Leftarrow \kappa \wedge \beta \Leftarrow \kappa]$;
(iii) $\{\forall\kappa[\alpha \Leftarrow \kappa \supset \neg\beta \Downarrow \kappa], \lambda \Leftarrow \beta, \neg\lambda \Leftarrow \alpha\}$ is satisfiable.

Proof. (i) P3.

(ii) This is trivial, because what is required to be proved is that $\exists\kappa[\beta \Leftarrow \kappa \wedge \alpha \Leftarrow \kappa]$ is deducible from $\exists\kappa[\alpha \Leftarrow \kappa \wedge \beta \Leftarrow \kappa]$.

(iii) Let $\mathbf{D}=\{d_1,d_2,d_3\}$, $\mathbf{E}=\varnothing$. Let r_1 be $\{\langle d_1,d_1\rangle,\langle d_2,d_2\rangle,\langle d_2,d_3\rangle,\langle d_3,d_2\rangle,\langle d_3,d_3\rangle\}$. Let $r_2=\varnothing$.

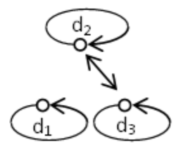

Figure 1. Counter-example to Barbara LML with all propositions descriptional

In Figure 1 unfilled circles represent intensions, arrows represent relations of inseparability. Since the only intension from which d_1 is inseparable is d_1 itself, and d_2 is compatible with d_1 (since $r_2=\varnothing$), d_2 is compatible with every intension from which d_1 is inseparable, so $\forall\kappa[\alpha \Leftarrow \kappa \supset \neg\beta \Downarrow \kappa]$ is satisfied. Since $\langle d_3,d_2\rangle \in r_1$, $\gamma \Leftarrow \beta$ is satisfied. Since $\langle d_3,d_1\rangle \notin r_1$, $\neg\gamma \Leftarrow \alpha$ is satisfied. All the postulates are satisfied: there are no incompatibilities, so P3, P4 and P5 are trivially satisfied; every intension is inseparable from itself; transitivity of inseparability is not violated; there are no quiddities, so P6 is trivially satisfied. This model is illustrated in Figure 1.

The model postulates three intensions, none of which is a quiddity. Each intension is inseparable from itself. The second and third are mutually inseparable. Every intension is compatible with itself and with the others. We could let the first intension be 'walking', the second and third 'humming'. Since 'humming' is compatible with every intension from which 'walking' is inseparable (viz. from 'walking' itself), 'Everything walking is possibly humming' is true. Since 'humming' is inseparable from 'humming', 'Everything humming is necessarily humming' is true. But 'Everything walking is necessarily humming' is not true, since 'humming' is not inseparable from 'walking'.

Now we come to Avicenna's substantial predications. What does Avicenna mean by the predicate's being 'perpetual [relative] to the existence of the substance of the subject'? I shall take this to mean (i) that the intension of the subject-term is a quiddity, and (ii) that the intension of the predicate-term is inseparable from every nature from which the intension of the subject-term is inseparable. Given this, we represent the La-proposition 'Every A is necessarily B' by the formula $Q\alpha \wedge \forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \beta \Leftarrow \kappa)]$. Bearing in mind that La- and Mo-propositions are contradictories, as are Le- and Mi-propositions, and that the La- and Le-propositions must entail the Li- and Lo-propositions, we can adopt the following representations for universal necessity- and possibility-propositions:

La: $Q\alpha \wedge \forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \beta \Leftarrow \kappa)]$
Le: $Q\alpha \wedge Q\beta \wedge \forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \beta \Downarrow \kappa)]$
Ma: $Q\alpha \wedge \forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \neg\beta \Downarrow \kappa)]$

Me: $Q\alpha \supset \forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \neg\beta \Leftarrow \kappa)]$.

Given these formal representations, we can prove:

T3. (i) *Le-conversion* is valid, (ii) *Li-conversion* is invalid, (iii) *Barbara LML* is valid – where all propositions are substantial.

What needs to be proved is that
(i) $Q\beta \wedge Q\alpha \wedge \forall\kappa[N\kappa \supset (\beta \Leftarrow \kappa \supset \alpha \Downarrow \kappa)]$ is deducible from
$Q\alpha \wedge Q\beta \wedge \forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \beta \Downarrow \kappa)]$;
(ii) $\{Q\alpha \wedge \exists\kappa[N\kappa \wedge \alpha \Leftarrow \kappa \wedge \beta \Leftarrow \kappa],\ \neg Q\beta \vee \neg\exists\kappa[N\kappa \wedge \beta \Leftarrow \kappa \wedge \alpha \Leftarrow \kappa]\}$ is satisfiable,
(iii) $Q\alpha \wedge \forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \gamma \Leftarrow \kappa)]$ is deducible from
$Q\alpha \wedge \forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \neg\beta \Downarrow \kappa)]$ and $Q\beta \wedge \forall\kappa[N\kappa \supset (\beta \Leftarrow \kappa \supset \gamma \Leftarrow \kappa)]$.

Proof. (i) Suppose $Q\alpha \wedge Q\beta$ and $\forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \beta \Downarrow \kappa)]$. Then $\forall\kappa[N\kappa \supset (\neg\beta \Downarrow \kappa \supset \neg\alpha \Leftarrow \kappa)]$. So, by P5, $\forall\kappa[N\kappa \supset (\beta \Leftarrow \kappa \supset \neg\alpha \Leftarrow \kappa)]$. So, by T1, $\forall\kappa[N\kappa \supset (\beta \Leftarrow \kappa \supset \alpha \Downarrow \kappa)]$. Hence, $Q\beta \wedge Q\alpha \wedge \forall\kappa[N\kappa \supset (\beta \Leftarrow \kappa \supset \alpha \Downarrow \kappa)]$.

(ii) Let **D**=$\{d_1,d_2\}$, **E**=$\{d_1\}$, where d_1 is assigned to α and d_2 is assigned to β. Since d_1 is the only member of **E**, **N**= $\{d_1\}$. Let r_1 be $\{\langle d_1,d_1\rangle, \langle d_2,d_1\rangle,\langle d_2,d_2\rangle\}$. Let $r_2 = \varnothing$. Since $d_1 \in$ **E** and $d_1 \in$ **N** and both $\langle d_1,d_1\rangle$ and $\langle d_2,d_1\rangle$ are in r_1,
$Q\alpha \wedge \exists\kappa[N\kappa \wedge \alpha \Leftarrow \kappa \wedge \beta \Leftarrow \kappa]$ is satisfied. Since
$d_2 \notin$ **E**, $\neg Q\beta \vee \neg\exists\kappa[N\kappa \wedge \beta \Leftarrow \kappa \wedge \alpha \Leftarrow \kappa]$ is satisfied. P1 is satisfied: r_1 contains both $\langle d_1,d_1\rangle$ and $\langle d_2,d_2\rangle$. So is P2: whenever $\langle d_i,d_j\rangle$ and $\langle d_j,d_k\rangle$ are in r_1 so is $\langle d_i,d_k\rangle$. P3-P5 are satisfied: r_2 is null. P6 is satisfied: d_1 is the only member of **E**, and $\langle d_1,d_1\rangle$ is in r_1.

A could be 'human', B 'capable of laughter'. The model then postulates that 'human' is an essential predicate from whose intension the intension of 'capable of laughter' is inseparable. 'Some human is necessarily capable of laughter' is true because the intension of 'human' is a quiddity and there is a nature (namely that quiddity) from which the intensions of both 'human' and 'capable of laughter' are inseparable. But 'Something capable of laughter is necessarily human' is not true, because the intension of 'capable of laughter' is not a quiddity.

(iii) Suppose $Q\alpha \wedge \forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \neg\beta \Downarrow \kappa)]$ and
$Q\beta \wedge \forall\kappa[N\kappa \supset (\beta \Leftarrow \kappa \supset \gamma \Leftarrow \kappa)]$. Then by T1, $Q\alpha \wedge \forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \beta \Leftarrow \kappa)]$.
So by P2, $Q\alpha \wedge \forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \gamma \Leftarrow \kappa)]$.

T3 agrees with Avicenna's results. He accepts the conversion of the Le-proposition (Dunyā, 1971, p.344). But he rejects Li-conversion, arguing that 'Something laughing is necessarily human' is true, since 'Everything laughing is necessarily human' is true, but 'Something human is necessarily laughing' is not true (Dunyā, 1971, p.335). And he regards Barbara LML as valid where the propositions are substantial. In a context of considering first-figure syllogisms with a possibility minor premise, he says:

> And if *All Bs are As necessarily* [sc. the major premise], then the truth is that the conclusion is necessary. (Street, forthcoming; Dunyā, 1971, p.393.)

T4. In the absence of P6, *Barbara LML* is invalid for substantial propositions.

What is required to be proved is that in the absence of P6 $\{Q\alpha \wedge \forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \neg\beta \Downarrow \kappa)], Q\beta \wedge \forall\kappa[N\kappa \supset (\beta \Leftarrow \kappa \supset \gamma \Leftarrow \kappa)], \neg Q\alpha \vee \neg\forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \gamma \Leftarrow \kappa)]\}$ is satisfiable.

Proof. Let **D**=**E**=$\{d_1,d_2,d_3\}$. Let r_1 be $\{\langle d_1,d_1\rangle, \langle d_2,d_2\rangle, \langle d_2,d_3\rangle, \langle d_3,d_2\rangle, \langle d_3,d_3\rangle\}$. Let r_2 be $\varnothing$.

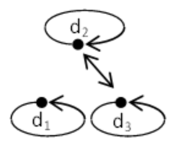

Figure 2. Counter-example to Barbara LML for substantial propositions, in the absence of P6

The model is illustrated in Figure 2, where filled circles represent quiddities, and arrows represent relations of inseparability.

Since $d_1 \in$ **E**, and every intension from which it is inseparable (namely d_1 itself) is inseparable from it, by D1 $d_1 \in$ **N**. Since $d_2 \in$ **E** and $d_3 \in$ **E** and every intension from which d_2 is inseparable (namely d_2 and d_3) is inseparable from it, $d_2 \in$ **N**. Similarly, $d_3 \in$ **N**. So **N**=**E**. $Q\alpha \wedge \forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \neg\beta \Downarrow \kappa)]$ is satisfied, because $d_1 \in$ **E** and d_2 is compatible with every nature from which d_1 is inseparable (since r_2 is null). $Q\beta \wedge \forall\kappa[N\kappa \supset (\beta \Leftarrow \kappa \supset \gamma \Leftarrow \kappa)]$ is satisfied, because $d_2 \in$ **E** and the only members of **N** from which d_2 is inseparable are d_2 and d_3, and d_3 is inseparable from both d_2 and d_3. $\neg Q\alpha \vee \exists\kappa[N\kappa \wedge \alpha \Leftarrow \kappa \wedge \neg\gamma \Leftarrow \kappa]$ is satisfied, because there is a nature from which d_1 is inseparable but from which d_3 is separable (viz. d_1).

Now for the postulates. P1 and P2 are satisfied, since the reflexivity and transitivity of inseparability are not violated. P3–P5 are satisfied trivially, since there are no incompatibilities. But P6 is not satisfied: there is a pair of quiddities which are not related by either inseparability or incompatibility, namely d_1 and d_2 (also d_1 and d_3).

Since Avicenna accepts Barbara LML, our analysis provides support for the view that he accepted P6.

Historians of Arabic logic have suggested that we should understand Avicenna's La-proposition to have an *ampliated* subject (e.g., Street, 2002, pp.129-160, 134). That is to say, the intension of the stated subject-term *A* should be understood to be 'compatible with *A*'. It turns out that our formalisation has the effect of ampliating the subject of the La-proposition in this fashion. Let us call a proposition with an ampliated subject-term an L'a -proposition; and let us represent such propositions formally as follows.

L'a: $Q\alpha \wedge \forall\kappa[N\kappa \supset (\neg\alpha \Downarrow \kappa \supset \beta \Leftarrow \kappa)]$.

We can then prove that the L'a-proposition is equivalent to the La-proposition. That the L'a-proposition entails the La-proposition is evident from P5. It remains to prove that the La-proposition implies the L'a.

T5. $Q\alpha \wedge \forall\kappa[N\kappa \supset (\neg\alpha \Downarrow \kappa \supset \beta \Leftarrow \kappa)]$ is deducible from $Q\alpha \wedge \forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \beta \Leftarrow \kappa)]$.

Proof.

1. $Q\alpha \wedge \forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \beta \Leftarrow \kappa)]$	[Assumption
2. $Q\alpha$	[1.
3. $\forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \beta \Leftarrow \kappa)]$	[1.
4. $\forall\kappa[N\kappa \supset (\neg\alpha \Downarrow \kappa \supset \beta \Leftarrow \kappa)]$	[2., 3., T1, D1
5. $Q\alpha \wedge \forall\kappa[N\kappa \supset (\neg\alpha \Downarrow \kappa \supset \beta \Leftarrow \kappa)]$	[2., 4.

4 Robert Kilwardby (d. 1279)

As a second example of a medieval logician for whom modal syllogistic is based on intensional considerations, I briefly mention the English Dominican Robert Kilwardby. In his commentary on Aristotle's *Prior Analytics* (Thom & Scott, 2016) Kilwardby distinguishes two types of necessity-proposition. One type requires merely that the predicate (or, what it signifies) be inseparable from the subject; the other makes a stronger claim, namely that there is a *per se* relationship between predicate and subject:

> ...propositions like this, which have the name of an accident as subject are not necessity-propositions *per se* but only incidentally. For a *per se* necessity-proposition requires the subject *per se* to be something under the predicate. But when it is said 'Every grammarian of necessity is a man', the subject is not *per se* something under the predicate. But it is granted to be necessary because 'grammarian' is not separated from that which is something under 'man'. (Thom & Scott, 2016, p.130.)

When Kilwardby speaks of what is 'under' the subject, he means not individual objects, but intensions. This is evident from the fact that he adopts what has been called the heterodox reading of the *dici de omni* principle (Malink, 2013, p.64). This is to say that he takes the statement 'All A is B' to be true only if for every common term C: if all C is A then all C is B (Thom & Scott, 2016, p.xxvii) Given this, something's being 'under' the subject-term involves a relation of inseparability between intensions.

We see from the quoted passage that according to Kilwardby the subject of a *per se* necessity-proposition cannot be an accidental term like 'grammarian'. He contrasts accidental with substantial terms:

> For every term is either substantial or accidental .... And I call those terms *substantial* which signify in the manner of an underlying subject, those *accidental* that do so in the manner of an accident belonging to something else. (Thom & Scott, 2016, p.418.)

A term's substantiality or accidentality is a matter of its signification, i.e. of its intension. A substantial term, unlike an accidental term, has a quiddity as its intension. Thus, we can represent what he calls 'incidental' universal affirmative necessity-propositions simply as statements to the effect that the intension of the predicate is inseparable from the intension of the subject (like Avicenna's descriptional necessities).

Per se necessity-propositions, on the other hand, reduce to what Aristotle calls *per se* predications:

> For necessary propositions reduce to some mode of *per se* inherence, following Aristotle's statement in *Posterior Analytics* I that 'Only *per se* inherences are necessary. (Thom & Scott, 2016, p.160.)

The reference is to Aristotle's *Posterior Analytics* I.6 74b12. According to the doctrine expounded in the early chapters of the *Posterior Analytics*, not only the subject but also the predicate of a *per se* predication belongs in the underlying subject's essence, being either a genus or a differentia of that subject. So Kilwardby is committed to requiring that both terms of an La-propositions have quiddities as their intensions. Given this, the La-proposition can be represented as $Q\alpha \wedge Q\beta \wedge \forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \beta \Leftarrow \kappa)]$.

Among *per se* necessity-propositions, Kilwardby makes a semantic distinction between universal affirmatives and universal negatives:

> A universal affirmative necessity affirms the predicate only of those things that are actually under the subject, not of those for which it's contingent to be under the subject. .... But it is otherwise with a negative necessity-proposition .... the predicate in that proposition is also actually denied, under a modality of necessity, of all things that are under the subject, and of all things for which it's contingent to be under the subject. (Thom & Scott, 2016, pp.514, 520.)

By 'being contingently under the subject' Kilwardby means 'being possibly under the subject'. He says that 'the possible and the contingent are convertible in all things' (Thom & Scott, 2016, p.100), meaning that there is a generic sense of 'contingent' which is equivalent to 'possible':

> ...the possible is not stated in the division of the contingent as a specific member of the contingent, but in order to designate a mode of the contingent, namely a mode of taking it in general as a genus. (Thom & Scott, 2016, p.146.)

I take him to be referring to those intensions that are compatible with the subject's intension. Thus the Le-proposition can be represented as $\forall\kappa[N\kappa \supset (\neg\alpha \Downarrow \kappa \supset \beta \Downarrow \kappa)]$.

In order to devise suitable formal representations of the Ma- and Me-propositions, we have to take into account Kilwardby's claim that in the case of *per se* necessity-propositions an Li-proposition 'Some A is necessarily B' implies the disjunction of the La- propositions 'Every A is necessarily B' and 'Every B is necessarily A':

> For among necessities, to state that the particular is true is the same as stating the universal, on account of the necessary relationship of the terms. (Thom & Scott, 2016, p.160.)

In order for this claim to be correct, the Li-proposition would have to entail that both of its terms have quiddities as their intension. For, if the Li-proposition had that entailment, then Kilwardby's claim about Li-proposition's entailing one or other of the corresponding La-propositions would be correct, provided we can assume P6. Suppose that the Li-proposition is true. Then $Q\alpha \wedge Q\beta$. But then, by P6, $\alpha \Downarrow \beta \vee \alpha \Leftarrow \beta \vee \beta \Leftarrow \alpha$. The first disjunct can be excluded, because the Li-proposition must at least entail $\exists\kappa[\alpha \Leftarrow \kappa \wedge \beta \Leftarrow \kappa]$; and if $\alpha \Downarrow \beta$ and $\beta \Leftarrow \kappa$ then by P4 $\alpha \Downarrow \kappa$, which by P5 entails $\neg\alpha \Leftarrow \kappa$. So we have $\alpha \Leftarrow \beta \vee \beta \Leftarrow \alpha$, which by P2 implies $\forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \beta \Leftarrow \kappa)] \vee \forall\kappa[N\kappa \supset (\beta \Leftarrow \kappa \supset \alpha \Leftarrow \kappa)]$, i.e. (since $Q\alpha \wedge Q\beta$) one or other of the La-propositions corresponding to our initial Li-proposition is true.

Notice, however, that we can arrive at the same conclusion even without P6. We could, for example, make the weaker assumption P7.

> P7. $(N\kappa \wedge Q\lambda \wedge Q\mu \wedge \lambda \Leftarrow \kappa \wedge \mu \Leftarrow \kappa) \supset (\lambda \Leftarrow \mu \vee \mu \Leftarrow \lambda)$.

P6 states that one of a pair of quiddities must be inseparable from the other if the two are compatible. P7 states that one must be inseparable from the other if both are inseparable from a common nature. P1–P5 provide no reason to prevent two quiddities being compatible even if there is no nature from which both are inseparable. Thus P7 is weaker than P6.

Suppose, then, that the Li-proposition is true. Then $Q\alpha \wedge Q\beta$. But then, by P7, $(N\kappa \wedge \alpha \Leftarrow \kappa \wedge \beta \Leftarrow \kappa) \supset (\alpha \Leftarrow \beta \vee \beta \Leftarrow \alpha)$. The Li-proposition must entail $\exists\kappa[N\kappa \wedge \alpha \Leftarrow \kappa \wedge \beta \Leftarrow \kappa]$. So, $\alpha \Leftarrow \beta \vee \beta \Leftarrow \alpha$, which by P2 implies $\forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \beta \Leftarrow \kappa)] \vee \forall\kappa[N\kappa \supset (\beta \Leftarrow \kappa \supset \alpha \Leftarrow \kappa)]$, i.e. (since $Q\alpha \wedge Q\beta$) one or other of the La-propositions corresponding to our initial Li-proposition is true.

Accordingly, I propose a minimal interpretation of Kilwardby in which P6 is dropped in favour of P7, and the Li-proposition is represented as $Q\alpha \wedge Q\beta \wedge \exists\kappa[N\kappa \wedge \alpha \Leftarrow \kappa \wedge \beta \Leftarrow \kappa]$. The Me-propositions must then be represented as $\neg Q\alpha \vee \neg Q\beta \vee \neg\exists\kappa[N\kappa \wedge \alpha \Leftarrow \kappa \wedge \beta \Leftarrow \kappa]$.

The Lo-propositions needs to follow from the Le-proposition. This will be the case if we represent it as $\exists\kappa[N\kappa \wedge \neg\alpha \Downarrow \kappa \wedge \beta \Downarrow \kappa]$ provided that we can assume $\exists\kappa N\kappa$. Accordingly I propose that Kilwardby requires an extra postulate:

P8. $\exists\kappa N\kappa$.

Given this representation of the Lo-proposition, the Ma-proposition must be represented as $\forall\kappa[N\kappa \supset (\neg\alpha \Downarrow \kappa \supset \neg\beta \Downarrow \kappa)]$.

La: $Q\alpha \wedge Q\beta \wedge \forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \beta \Leftarrow \kappa)]$
Le: $\forall\kappa[N\kappa \supset (\neg\alpha \Downarrow \kappa \supset \beta \Downarrow \kappa)]$
Ma: $\forall\kappa[N\kappa \supset (\neg\alpha \Downarrow \kappa \supset \neg\beta \Downarrow \kappa)]$
Me: $\neg Q\alpha \vee \neg Q\beta \vee \neg\exists\kappa[N\kappa \wedge \alpha \Leftarrow \kappa \wedge \beta \Leftarrow \kappa]$.

Given these formalisations, and assuming that all necessity-propositions are *per se*, we can prove the following theorem:

T6. (ii) *Le-conversion* is valid, (iii) *Li-conversion* is valid, (vi) *Barbara LML* is invalid.

What is required to be proved is:
(i) $\forall\kappa[N\kappa \supset (\neg\beta \Downarrow \kappa \supset \alpha \Downarrow \kappa)]$ is deducible from $\forall\kappa[N\kappa \supset (\neg\alpha \Downarrow \kappa \supset \beta \Downarrow \kappa)]$;
(ii) $Q\beta \wedge Q\alpha \wedge \exists\kappa[N\kappa \wedge \beta \Leftarrow \kappa \wedge \alpha \Leftarrow \kappa]$ is deducible from $Q\alpha \wedge Q\beta \wedge \exists\kappa[N\kappa \wedge \alpha \Leftarrow \kappa \wedge \beta \Leftarrow \kappa]$;
(iii) $\{\forall\kappa[N\kappa \supset (\neg\alpha \Downarrow \kappa \supset \neg\beta \Downarrow \kappa)], Q\beta \wedge Q\gamma \wedge \forall\kappa[N\kappa \supset (\beta \Leftarrow \kappa \supset \gamma \Leftarrow \kappa)], \neg Q\alpha \vee \neg Q\gamma \vee \neg\forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \gamma \Leftarrow \kappa)]\}$ is satisfiable.

Proof.

(i) Trivial.
(ii) Trivial.
(iii) We can re-use the model that was used in **T4**. See Figure 2. Let **D**=**E**= $\{d_1, d_2, d_3\}$. Let r_1 be $\{\langle d_1, d_1\rangle, \langle d_2, d_2\rangle, \langle d_2, d_3\rangle, \langle d_3, d_2\rangle, \langle d_3, d_3\rangle\}$. Let $r_2 = \varnothing$. By D1, **N**= $\{d_1, d_2, d_3\}$.

This model satisfies $\forall\kappa[N\kappa \supset (\neg\alpha \Downarrow \kappa \supset \neg\beta \Downarrow \kappa)]$, because every nature is compatible with d_1 and with d_2, since r_2 is null. The model satisfies $Q\beta \wedge Q\gamma \wedge \forall\kappa[N\kappa \supset (\beta \Leftarrow \kappa \supset \gamma \Leftarrow \kappa)]$, because $d_2 \in$ **E** and $d_3 \in$ **E** and d_3 is inseparable from every nature from which d_2 is inseparable (namely, d_2 and d_3). And the model satisfies $\neg Q\alpha \vee \neg Q\gamma \vee \neg\forall\kappa[N\kappa \supset (\alpha \Leftarrow \kappa \supset \gamma \Leftarrow \kappa)]$, because there is a nature (d_1) from which d_1 is inseparable and d_3 is separable (since $\langle d_3, d_1\rangle$ is not a member of r_1).

T6 agrees with Kilwardby's results. He accepts Le-conversion and Li-conversion:

> First [Aristotle] describes the conversion of necessity-propositions, saying that they convert just like assertorics. For the universal negative converts in its terms without qualification; but the universal affirmative and the particular affirmative convert to a particular affirmative. (Thom & Scott, 2016, p.122.)

(He thinks Aristotle is right.) And here he is on the invalidity of Barbara LML:

> Now, the reason why the necessity, or even the assertoric, doesn't follow is this. A universal affirmative necessity affirms the predicate only of those things that are actually under the subject, not of those for which it's contingent to be under the subject. For the proposition 'Every man of necessity is an animal' doesn't say that whatever can be a man is an animal, but whatever is a man is an animal. And accordingly when a proposition of this type is stated as major premise and the minor is a contingency, an actual affirmation of those things which are taken contingently under the middle does not follow. And accordingly neither an affirmative necessity nor an affirmative assertoric follows, because in both cases the predicate is actually affirmed of the subject. (Thom & Scott, 2016, p.520.)

5 Leibniz

Returning now to Leibniz, we prove:

T7. The Leibniz-Glashoff analysis of (non-modal) *a*- and *e*-predications is equivalent to our analysis of Avicenna's descriptional necessity-propositions and Kilwardby's incidental necessity-propositions.

Proof. The Leibniz-Glashoff analysis of 'Every A is B' is 'Everything positively contained in the predicate is positively contained in the subject, and everything negatively contained in the predicate is negatively contained in the subject'. The analysis of 'No A is B' is 'Something positively contained in one term is negatively contained in the other'. Taking positive containment as the converse of inseparability, and negative containment as incompatibility, the Leibniz-Glashoff truth-condition for *All A is B* becomes:

> LA. $\forall\kappa[\kappa \Leftarrow \beta \supset \kappa \Leftarrow \alpha] \wedge \forall\kappa[\kappa \Downarrow \beta \supset \kappa \Downarrow \alpha]$.
> By P1, LA implies $\beta \Leftarrow \alpha$.
> By P2, $\beta \Leftarrow \alpha$ implies $\forall\kappa[\kappa \Leftarrow \beta \supset \kappa \Leftarrow \alpha]$; and by P4, $\beta \Leftarrow \alpha$ implies $\forall\kappa[\kappa \Downarrow \beta \supset \kappa \Downarrow \alpha]$.

The Leibniz-Glashoff truth-condition for *No A is B* is

> LE. $\exists\kappa[\kappa \Leftarrow \alpha \wedge \kappa \Downarrow \beta] \vee \exists\kappa[\kappa \Leftarrow \beta \wedge \kappa \Downarrow \alpha]$
> By P3 and P4, LE implies $\alpha \Downarrow \beta$.
> By P1, $\alpha \Downarrow \beta$ implies LE.

In this light we can say that the analysis Leibniz gives of categorical propositions (according to Glashoff) is equivalent to the meaning Avicenna assigned to descriptional necessity-propositions and Kilwardby to incidental necessity-propositions. But the more sophisticated analyses developed by the medievals—the substantial necessity-propositions of Avicenna and the *per se* necessity-propositions of Kilwardby—find no parallel in Leibniz. At the same time, Leibniz's mathematical representation of categorical propositions, to the best of my knowledge, has no precedent in the medieval logicians.[1]

[1] I thank Tony Street for his generous help with the Avicenna material.

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