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8 On some French probabilists of the twentieth century: Fréchet, Borel, Lévy

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Abstract. In the first half of the Twentieth century a number of authors active in distant parts of Europe and in different areas of scientific research shared a probabilistic approach to science and knowledge in general, albeit embracing different interpretations of probability. My "Probabilistic Epistemology: a European Tradition"[1] focussed on the work of Polish logician Janina Hosiasson, British mathematician Frank Plumpton Ramsey and geophysicist Harold Jeffreys, Italian statistician Bruno de Finetti, and German philosopher of science Hans Reichenbach, arguing that one can speak of a European tradition in probabilistic epistemology.

To clarify the matter, by *probabilistic epistemology* I mean the view that probability is an essential ingredient of knowledge, and that induction is a fundamental component of the scientific method. Such a view is grounded in the conviction that certainty of knowledge and completeness of information are unachievable, and as Patrick Suppes clearly stated, "it is the responsibility of a thoroughly-worked-out empiricism to include an appropriate concept of uncertainty at the most fundamental level of theoretical and methodological analysis. Probabilistic methods provide a natural way of doing so" (Suppes, 1984, p. 99).

The purpose of this paper is to expand on my earlier work by adding to the picture the French mathematicians Maurice Fréchet, Émile Borel and Paul Lévy, all of whom advocated a probabilistic approach to epistemology, bringing new evidence that probabilistic epistemology was widespread throughout Europe in the first half of the

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[1] See Galavotti (2014).

Twentieth century. Apart from that, the philosophy of probability embraced by such outstanding mathematicians seems worthy of attention in itself. Given the scant literature on the topic, confined to its technical aspects, the present analysis will broaden the picture of the debate on the foundations of probability.

Keywords: probability, epistemology, frequentism, subjectivism.

1 European probabilism in the first half of the twentieth century

Before addressing the philosophy of probability of the French authors, I will briefly sketch the main traits of the position embraced by Richard von Mises, Hans Reichenbach, and Bruno de Finetti to give some substance to criticism moved against these authors by the French probabilists examined in the second part of the paper.

1.1 Richard von Mises' frequentism

Richard von Mises (1883-1953) gave great impulse to the debate on the foundations of probability in the first decades of the last century. Evidence of this is the fact that right at the beginning of the opening lecture (*conference d'introduction*) delivered at the "Colloque consacré a la théorie des probabilités" held in 1937 in Geneva, Maurice Fréchet credits von Mises with "having awakened interest in questions previously addressed in a fragmentary way" (Fréchet, 1938b, p. 19).

Von Mises is deemed the main representative of the frequency theory of probability. According to this view, probability is a characteristic of phenomena that can be empirically analysed by means of observed frequencies. It is defined as the limit of the relative frequency of a given attribute observed in the initial part (sample) of an indefinitely long sequence of repeatable events. A key tenet of this interpretation is that probability values are in general unknown, but can be approached by means of frequencies, as the number of observed elements increases.

The core of von Mises' theory is the notion of *collective* defined as follows: "A collective is a mass phenomenon or a repetitive event, or, simply, a long sequence of observations for which there are sufficient reasons to believe that the relative frequency of the observed attribute would tend to a fixed limit if the observations were indefinitely continued. This limit will be called the probability of the attribute considered within the given collective" (von Mises, 1951/1957, p. 15). In order to qualify as a collective, a sequence has to: (1) be indefinitely long, (2) exhibit frequencies that tend to a limit, and (3) be random. Randomness rests on the method of "place selection", which consists in extracting sub-sequences from the original sequence by considering only the place that each member occupies in the sequence, while ignoring their distinctive properties. Each place selection is defined by a rule that states for any element of the sequence whether it ought to be included in the sub-sequence or not. For instance, the sub-sequence obtained by picking all members whose place number in the sequence is a prime number would satisfy the place selection method. Von Mises defines randomness as *insensitivity to place selection*, which obtains when "the limiting values of the relative frequencies in a collective must be independent of all possible place selections" (von Mises, 1951/1957, p. 25).

Having defined collectives along these lines, von Mises proceeds to formulate the principles of probability theory in terms of collectives by means of the operations of selection, mixing, partition and combination. Adoption of this conceptual machinery is intended to secure probability a foundation that is both empirical and objective. The author maintains that probability applies only to collectives, as suggested by the title of one section of the first chapter of *Probability, Statistics and Truth*: "First the collective - then the probability" (von Mises, 1951/1957, p. 18). A major drawback of von Mises' position is that it makes no sense to apply probability to single events. He openly admitted this, deeming single-case probability simply meaningless.

A remarkable feature of von Mises' perspective is that it embodies a probabilistic epistemology open to indeterminism. The idea is that the developments brought into physics by quantum mechanics have imposed the need to ground the whole edifice of science on a statistical conception of nature, granting indeterminism the same plausibility traditionally attached to determinism. Deeply convinced that probability and statistics offer the most powerful heuristic tool for investigating reality, von Mises heralds a probabilistic approach to the building of scientific knowledge.[2] Such a conviction inspires the closing passage of *Probability, Statistics and Truth*, where the author writes that "starting from a logically clear concept of *probability*, based on experience, using arguments which are usually called *statistical*, we can discover *truth* in wide domains of human interest" (von Mises, 1951/1957, p. 220).

Much of the debate on von Mises' work has focussed on the notion of randomness. After criticism advanced by a number of authors including Alonzo Church, Abraham Wald, Jean Ville, Arthur Copeland and others, von Mises' unrestricted notion of randomness was abandoned in favour of a weaker notion entailing a restricted domain of place selections. Another problematic aspect of von Mises' version of frequentism is the impossibility of applying probability to single events, for it is undeniable that in many areas, including everyday life and the social sciences, the need to speak of single-case probabilities is widely felt. The same holds for Quantum Mechanics, where one speaks of single atoms, particles, and so on; it was precisely in order to solve this problem that in the 50s of the last century Karl Popper put forward the so-called propensity interpretation of probability.[3]

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[2]See von Mises (1951/1968) for more on von Mises' probabilistic epistemology.

[3]See Galavotti (2005) for more on the propensity theory, as well as on the authors dealt with in the first part of this paper.

1.2 Hans Reichenbach's frequentism

Hans Reichenbach (1891-1953) worked out a version of frequentism that strays in various ways from von Mises'. In particular, Reichenbach sets himself the task of developing a flexible theory suitable for wide applicability, in which it makes sense to talk of single-case probabilities. In addition, Reichenbach adopts a notion of randomness limited to a restricted domain of selections, together with a notion of "practical limit" relative to "sequences that, in dimensions accessible to human observation, converge sufficiently and remain within the interval of convergence"(Reichenbach, 1949/1971, p. 347). He also develops a theory of induction, together with an argument for its justification.

Reichenbach shares von Mises' conviction that it is probability, not truth, that should be put at the core of a sound reconstruction of scientific knowledge, because "the ideal of an absolute truth is an unrealizable phantom" (Reichenbach, 1937, p. 90). Stress on action, prediction, and practice confers a pragmatist flavour to Reichenbach's view of knowledge: he explicitly acknowledges his debt towards Charles Sanders Peirce and William James in connection with the theory of meaning, which revolves around the tenet that "there is as much meaning in a proposition as can be utilized for action" (Reichenbach, 1938, p. 80).

Reichenbach names *Rule of induction* the canon by which probability is obtained as the limit of observed frequency, and emphasizes that any probability attribution is a *posit*, namely "a statement with which we deal as true, although the truth value is unknown" (Reichenbach, 1949/1971, p. 373). Posits differ depending on whether they occur within primitive or advanced knowledge. Reichenbach calls *primitive* the state of knowledge in which no prior knowledge of probabilities is available, and *advanced* the state of knowledge in which prior probabilities are available. In the first case, the Rule of induction yields prior probabilities, and posits are called *blind*; in the second, the probability calculus can be used to combine prior probabilities, and the posits so obtained are called *appraised*. The interplay between blind and appraised posits generates *the method of concatenated inductions*, which is intrinsically *self-corrective*, because the Rule of induction guarantees convergence of probability estimates made by its means.

The self-corrective character of the method of concatenated inductions lies at the core of Reichenbach's pragmatic justification of induction: inductive inference, and more precisely the Rule of induction, is justified because it provides the best possible guide to the future. As Reichenbach wrote: "it is a method of which we know that if it is possible to make statements about the future we shall find them by means of this method" (Reichenbach, 1949/1971, p. 475).

A major point of divergence with von Mises is the fact that Reichenbach does not consider single-case probabilities meaningless, and he attempts to accommodate them within the frequentist outlook. The idea is that posits regarding single events receive a weight from the probabilities attached to the reference class to which the event in question has been assigned. The crucial issue here is the choice of the reference class, which must obey a requirement of *homogeneity*. A reference class is homogeneous if it includes all the properties that are taken to be relevant to the event under study. This is obviously a very strong requirement, hardly ever met in practice because - apart from cases falling directly under the scope of scientific theories - one can never be sure that all relevant information has been taken into account. As a matter of fact, this requirement clashes with Reichenbach's intent to develop a version of frequentism suited to a wide range of applications both in science and everyday life.[4]

1.3 Bruno de Finetti's subjectivism

Starting from the late 1920s, Bruno de Finetti (1906-1985) developed a radical form of probabilism that can be described as a blend of pragmatism and the kind of empiricism that is today called anti-realism. It moves from a rejection of the notions of truth, determinism and "immutable and necessary" laws, to reaffirm a conception of science as a product of human activity, deeply imbued with probability. De Finetti identifies the aim of science with prediction, and regards (subjective) probability as the best possible tool for making good forecasts.

According to the subjective interpretation, probability is a quantitative expression of the degree of belief in the occurrence of an event, entertained by a person in a state of uncertainty. It is taken as a primitive notion having a psychological foundation, which requires an operative definition in order to be measured and used in practice. A well-known method for measuring degrees of belief is the betting scheme, according to which probability expresses the conditions under which someone is ready to bet on the occurrence of an event. This method, which is endowed with a long tradition dating back to the Seventeenth century, is deemed by Frank Plumpton Ramsey - the other "father" of the subjective interpretation - "fundamentally sound" although "insufficiently general" and "necessarily inexact",[5] because it suffers from problems such as the diminishing marginal utility of money, and the personal (greater or less) disposition to gambling. In view of this, both Ramsey and de Finetti stressed that subjective probability can be given an operational definition also by means of other methods: Ramsey adopted a system of preferences, and de Finetti turned to penalty methods like Brier's rule.[6] The cornerstone of the subjective theory is the notion of coherence, for as Ramsey and de Finetti showed, coherent degrees of belief obey the rules of additive probabilities. Put otherwise: the laws of probability can be derived from the assumption of coherence.

All coherent probability functions are admissible for upholders of the subjective theory. This means that once coherence is guaranteed disagreement is admitted. In other

[4]For more on Reichenbach's position see Galavotti (2011a).

[5]See Ramsey's "Truth and Probability" in Ramsey (1931), reprinted in Ramsey (1990).

[6]See de Finetti (1970/1975).

words, for subjectivists probability evaluations are not univocally determined by evidence, and the estimation of probability also depends on subjective elements such as experience and personal abilities, in addition to empirical evidence. With increasing evidence, though, the opinions of different people will converge. The result, known as "de Finetti's representation theorem" (although the author always refused to call it a "theorem"), proves that convergence between subjective probability and observed frequencies is assured by the adoption of exchangeability in connection with Bayes' rule.

Having said that, it must be added that for de Finetti it does not make sense to regard the Bayesian method - or the inductive method, which for him amounts to the same thing - as self-corrective. Bayesian reasoning entails updating probability evaluations in the light of new evidence, but updating should not be construed as approximation to true probabilities. De Finetti entrusted this conviction to the statement "Probability does not exist", printed in capital letters in the Preface of the English edition of *Theory of Probability*. Such a statement, often mentioned and just as often misunderstood, has been taken to imply that all coherent probability evaluations are on a par. This is not so: albeit rejecting as metaphysical the idea that probability is an objective property of phenomena, de Finetti took very seriously the problem of the objectivity of probability evaluations, namely the problem of devising methods that enable successful predictions to be made. In this spirit, he maintained that both "(1) the objective component, consisting of the evidence of known data and facts; and (2) the subjective component, consisting of the opinion concerning unknown facts based on known evidence" (de Finetti, 1974, p. 7) are essential ingredients of probability evaluations.

2 The French milieu

This section of the paper tackles the philosophy of probability of the French mathematicians Fréchet, Borel and Lévy, all of whom gave important contributions to various branches of mathematics and the theory of probability.[7] In addition, they were actively involved in the lively debate on the foundations of probability carried on in a number of publications and congresses in the first half of the twentieth century. Particularly important in that connection was the *International Congress of Mathematicians* that took place in Geneva in 1937. The meeting hosted a *Colloque consacré a la théorie des probabilités* that brought together a number of renowned mathematicians, statisticians and scientists, including among others Bruno de Finetti, William Feller, Maurice Fréchet, Richard von Mises, Werner Heisenberg, Eberhard Hopf, Jerzy Neyman, George Polya, J.F. Steffensen, Francesco Cantelli and Abraham Wald. Another important gathering was the *XVIII Congrès international de philosophie des sciences* held in Paris in 1949, which included one session entitled *Calculus*

[7] See von Plato (1994) for the contribution given by Borel, Fréchet and Lévy to the theory of probability.

of probabilities, where some of the most outstanding probabilists of the time, including Maurice Fréchet, Émile Borel, Bruno de Finetti, Jerzy Neyman, Jean Ville and Paul Lévy, delivered papers devoted to technical as well as philosophical aspects of the notion of probability. The proceedings of both of these conferences were published in the periodical *Actualités scientifiques et industrielles*. Also worth mentioning is a monographic issue of the journal *Dialectica* published in 1949 under the title *The probable knowledge*, collecting, among others, papers by Émile Borel, Bruno de Finetti, George Polya, Corrado Gini, Paul Lévy, Padrot Nolfi, M.S. Bartlett, and Subrahmanyan Chandrasekhar.

2.1 Maurice Fréchet's "modernized axiomatic theory"

Professor of general mathematics and the calculus of probabilities at the Sorbonne University in Paris, Maurice Fréchet (1978-1973) is considered the founder of the theory of abstract spaces, and gave outstanding contributions to topology and functional analysis. He organized a series of lectures at the École Normale Superieure and at the Institut Poincaré, where in 1935 Bruno de Finetti was invited to deliver the lecture course later published under the title "La prévision: ses lois logiques, ses sources subjectives". Although he did not share de Finetti's subjective approach, Fréchet thought highly of him, and the two entertained a correspondence in the course of which, among other things, Fréchet called de Finetti's attention to the work of Ramsey, pointing out the similarity between their views on probability.[8] Moreover, it was Fréchet who in 1939 suggested de Finetti adopt the term "exchangeability" instead of "equivalence", which he had used until then.[9]

Fréchet is deeply convinced that the notion of probability should be addressed from the standpoint of its applications. To accomplish that task the mathematics of probability cannot do the whole job, and must be backed up by philosophy. The idea is that probability should provide a guide to action and decision, and hence must be applicable to the problems encountered in all sorts of practical situations. Moreover, probability should be amenable to "verification" by observing the success of predictions made by its means. Fréchet endorses Augustin Cournot's tenet that the application of probability to real phenomena requires going beyond mathematics, and quoting a passage pointed out to him by Paul Lévy states that "in order to go from the idea of an abstract relationship to that of a law that can be useful in the realm of phenomena, *mathematical reasoning* [...] *is obviously insufficient*. One needs to appeal to other notions, to other principles of knowledge; in a word: *one needs philosophical criticism*" (Fréchet, 1938a, p. 43). According to Fréchet, a similar attitude was shared by Henri Poincaré

[8]See Box 6 of *Bruno de Finetti Collection; Archives of Scientific Philosophy*, Hillman Library of the University of Pittsburgh.

[9]This is mentioned by de Finetti in his "farewell lecture" at the University of Rome, see de Finetti (1976, p. 283).

and Francesco Cantelli. Emphasis is on the need to build a bridge between the abstract, mathematical theory of probability and its applications, or between a schema representing reality and the corresponding elements of reality.

Fréchet does not regard the problem as peculiar to probability theory, but shared by all the empirical sciences. In this spirit, he thinks that the evaluation of probability is similar to the measurement of a physical magnitude, and such an analogy plays a crucial role in his conception of probability, labelled a *modernized axiomatic theory*. His theory revolves around the tenet that probability is to be construed "as a physical magnitude attached to an event and a category of trials, of which the frequencies of the event in a great number of trials are approximated measures" (Fréchet, 1951, p. 5). When dealing with probability one proceeds "exactly as in the experimental sciences where measures are *generally* approximate values of physical magnitudes" (Fréchet, 1939-1940, p. 12), the difference being that in probability, as opposed to other disciplines, the precision of measurement increases as the number of observations grows. This process results from a combination of empirical and axiomatic considerations, in which a fundamental role is played by what Fréchet calls *inductive synthesis*. In order to accomplish an inductive synthesis one proceeds as follows: "The axiomatic theory aims at - and allows - certain unknown probabilities *p* to be derived from other known probabilities *p*'. The interpretation of probability allows each *p*' to be calculated approximately on the basis of the observation of certain frequencies *f*' and to put each *p* approximately equal to a frequency *f*. In the end, one will succeed in calculating some frequencies, whose direct observation could be impossible or difficult, starting from other frequencies *f*' which are easily observed" (Fréchet, 1946, p. 150).

The peculiarity of the procedure Fréchet describes is that the application of the axioms should be preceded by the act of verifying whether in practical situations the axioms apply to the particular class of events under study. In that sense, the measurement of probability requires a synthesis between the axiomatic theory and practical situations. According to Fréchet, "In this synthesis intuition and contact with reality are the main directions to follow and therefore rigour is not supreme. Applied to probability, this leads us to conclude from the practical statistical processes that any frequency is to be considered an approximate measure of one *physical constant* attached to an event E and to a category C of trials" (Fréchet, 1939-1940, p. 11). The inductive synthesis so described is the cornerstone of the interpretation of probability heralded by Fréchet, and is meant to represent the element of novelty implicit in the author's expression "modernised axiomatic theory". As repeatedly emphasized in his writings, Fréchet's interpretation has a *practical* import, and falls outside the axiomatic theory to which it is linked by means of inductive synthesis.

Obviously, the crucial step in such a synthesis is the choice of the proper "category of trials". In this regard, Fréchet faces a reference class problem similar to that besetting Reichenbach's attempt to apply the frequency theory to single case evaluations. Fréchet admits that such a choice depends on subjective elements. However, wanting to retain an objective notion of probability, he emphasizes that the subjective elements neither concern nor affect the probability itself, but rather the way a certain problem is formulated in a given context. It is at that stage that the choice of the reference class is made, depending on the information available and the purpose being pursued. Fréchet observes that such a choice "cannot go beyond our knowledge, but it could use only part of it; an act of volition [...] *decides* the choice of the category of trials" (Fréchet, 1938a, p. 50). It is the purpose for which probability is being evaluated that guides the choice of the reference class, after deciding which of all the known features of a given phenomenon are deemed relevant and picked as characterizing the category of trials. Fréchet emphasizes that the choice of the reference class has nothing to do with probability, which "comes into play after the event and the category of trials have been chosen, at which point it is entirely determined and its value is independent of the person who made the choice" (Fréchet, 1938a, p. 50).

Fréchet's answer to the question whether probability is objective or subjective is that "what is subjective is not the *value* of probability, it is the *problem* that is being posed, namely the choice that is made of the category of trials in connection to which one calculates the probability of an event" (Fréchet, 1938a, p. 50). To the subjective interpretation of probability Fréchet opposes the view that "all of my information comes into play with the choice of the category of trials, namely in connection with the specification of the problem, not with its solution" (Fréchet, 1946, p. 146). The probability of an event must be kept separate from someone's degree of belief in its occurrence in the same way as "a distinction is to be made between the correct solution to a problem and the solution given by some student, which can vary from one student to another" (Fréchet, 1946, p. 143).

In that regard, Fréchet claims agreement with Harold Jeffreys' tenet that probability values are univocally determined by evidence and that one must distinguish between "objective probability and subjective degrees of belief, which are but personal estimates of the same" ((Fréchet, 1946, p. 146), footnote 2).[10] Fréchet emphasizes that in order to approach the value of probability of an event, subjective estimates "must be corrected", and this can only be done by recourse to observed frequencies. Therefore "those probabilists who by assimilating probabilities with degrees of belief thought that they could avoid appealing to frequencies, will be forced to turn to them when asked to justify the success of their interpretation" (Fréchet, 1946, p. 145). This, and other assertions, such as the claim that subjectivists "are just happy with grading a state of mind" (Fréchet, 1951, p. 17) suggest that Fréchet fell into the misunderstanding pointed out in Section 2.3, by which upholders of the subjective interpretation fix probability values without taking frequencies into account. As already observed, this is the result of a misrepresentation of the subjective approach. An additional criticism against subjectivism concerns the notion of coherence, to which Fréchet objects that most people do not behave coherently. Instead of "rational" coherent degrees of belief, most of the time they retain "irrational probabilities" that do not satisfy coherence. Therefore, to Fréchet's eyes the subjective interpretation of Savage and de

[10]See Galavotti (2003) for more on Jeffreys' views on probability and probabilistic epistemology.

Finetti refers to a *fictitious* agent, not to the way in which real agents behave in practice (see Fréchet, 1954).

Albeit sharing his objective approach to probability, Fréchet is also critical of Reichenbach. The crucial difference between their perspectives is that Fréchet does not impose a homogeneity requirement on the choice of the reference class, leaving it to the context. Fréchet stresses that any problem can be tackled from different angles, and therefore there is no one single reference class to be taken as suitable in a given situation. Borrowing an example from Borel, he discusses the probability of death of an individual, observing that it has a different meaning for his doctor and for his insurance company, so that the first will choose the population of "people of the same age, same weight, same blood pressure, same lung capacity, etc." while the insurance company will consider the population of insured having the same age. "The values of the corresponding probabilities can differ without ceasing to be compatible. [...] the probabilities evaluated by the doctor and the insurance company can be assimilated to two distinct physical magnitudes. Their approximate measure will be obtained by means of statistics properly made but obtained from different populations" (Fréchet, 1938a, pp. 50–51).

Fréchet also raises a more general objection to the frequency interpretation of von Mises and Reichenbach, observing that "those who advocate the 'proceed to the limit' definition think that they have in this way made the theory nearer practice. In fact, they have made it more remote." (Fréchet, 1939-1940, p. 17). To substantiate this claim, Fréchet borrows an example from Bruno de Finetti: take an unlimited sequence of trials where the frequency of Tails observed in n trials tends to $1/2$ when n increases, but in which one has obtained Tails 10,000 times. This could be a collective, in which the probability of Tails is $1/2$, however such a probability would not be $1/2$ for whatever experimenter (see Fréchet, 1939-1940, pp. 17–18). Although frequencies are the essential ingredient for estimating probabilities, for Fréchet estimates are not to be obtained by proceeding to the limit as the number of observed cases increases. Frequency is not part of the definition of probability, it is rather the tool that allows probability values to be approximated in practice. Fréchet repeatedly emphasizes that his position is a *practical interpretation* according to which frequency is an "empirical measure of probability" (Fréchet, 1951, p. 12). Unlike von Mises' interpretation which includes "a measure of probability in the axiomatic theory" (Fréchet, 1951, p. 12) and embodies a constructive definition of probability, Fréchet's viewpoint is meant as a descriptive account falling outside the axiomatic theory.

To conclude, Fréchet's major concern is that probability must provide a guide to action and decision. To accomplish that task it must be applicable to problems encountered in science as well as in practical situations occurring in everyday life. Such a concern imbues both Fréchet's concept of probability and the criticism he moves against frequentism and subjectivism.

2.2 Émile Borel's moderate subjectivism

A leading mathematician, Émile Borel (1871-1956) gave substantial contributions to analysis and group theory, did seminal work in probability theory, and developed an original philosophy of probability. Between 1925 and 1939 he published a series of monographs under the collective title *Traité du calcul des probabilités et ses applications*, ending with the essay *Valeur pratique et philosophie de la probabilité*, where his views on the nature of probability are spelled out with great clarity.

Borel takes a probabilistic attitude towards science and knowledge in general: "probability lies at the core of scientific knowledge" since "the value of all scientific results can be assessed only by means of a probability coefficient" (Borel, 1939/1952, p. 10). The theory of probability is of paramount importance because "not only does it possess the same practical and philosophical value of all other scientific theories, but it is the basis of all our knowledge" (Borel, 1939/1952, p. 11). Borel was not only a successful scientist but also a politician and a social reformer - among other things he was for twelve years a member of the Chamber of Deputies and in 1925 served as minister of the Navy.

Convinced that probability is also of great value for life and that "the practical value of probability can surpass that of the rest of human knowledge" (Borel, 1939/1952, p. 42), Borel was actively engaged in pedagogical reforms aimed at educating people on probability from their youth, because in this way "one will reduce the persistence of many prejudices" (Borel, 1939/1952, p. 10). He struggled to counteract the widespread resistance to thinking in probabilistic terms, and in an effort to convince people that "there are only statistical truths" (Borel, 1907/2014, p. 1083) argued that "the mathematical answer to be given to many practical questions is a coefficient of probability. Such an answer will not seem satisfactory to many minds, who expect certainty from mathematics. This is a very bad inclination; it is utterly regrettable that the education of the public is, in this respect, so little advanced; this may be due to the fact that the mathematics of probability remains a subject of near universal ignorance, even though every day it intrudes a bit more into everyone's life (various insurance policies, mutual aid societies, retirement pensions, etc.). A coefficient of probability constitutes a perfectly clear answer, corresponding to an absolutely tangible reality. Some minds will maintain that they 'prefer' certainty; they might as well 'prefer' that 2 plus 2 were 5" (Borel, 1907/2014, p. 1087). This long passage has been reported here as evidence of Borel's deep concern for the theoretical and practical value of probability in science and life too.

To start with, Borel calls attention to the fact that the theory of probability differs from other sciences, and from other branches of mathematics as well. For one thing, other magnitudes - including numbers belonging to arithmetic - are measured by means of a measurement unit that can vary, and according to the chosen unit they can be said to be large or small. This is not so for probability where the unit of measurement is unique, and by convention probability 1 equals certainty. Therefore "we have an absolute scale for measuring the degree of smallness of probabilities" (Borel, 1939/

1952, p. 6). In addition, the theory of probability differs from the other sciences because of the nature of its object, given that "by its very nature" probability "cannot claim to give us certainties" (Borel, 1939/1952, p. 4). While from an abstract or axiomatic point of view, once some principles have been postulated, probability values are well defined, when one moves from theory to practice the import of such values is uncertainty "precisely because the very object of deductions and calculations is simply a probability, not a number, a length, a time, as in Arithmetic, Geometry, Astronomy" (Borel, 1939/1952, p. 5).

For Borel, probability is endowed with a different value depending on the body of information available within a given context. Typically, probability has a more objective meaning in science, where its assessment is grounded on a strong body of information shared by the community of scientists. By contrast, probabilities attached to individual judgments can have "different values for different individuals" (Borel, 1924/1964, p. 50). In a review of Keynes' *Treatise on Probability* originally published in 1924 Borel criticizes Keynes for concentrating only on the probability of judgments, overlooking the application of probability to science. Incidentally, he attributes such an attitude to English rather than continental authors who are deemed more attentive to the progress of science, especially physics. As Borel emphasizes: "the probability that an atom of radium will explode tomorrow is, for the physicist, a constant of the same kind as the density of copper or the atomic weight of gold. Albeit these constants are always at the mercy of the progress of physical-chemical theory, they are constants in the present state of science" (Borel, 1924/1964, p. 50).

Borel does not share Keynes' tenet that there are probabilities which cannot be evaluated numerically. In that connection he rather agrees with subjectivists, with whom he also shares the conviction that "the method of betting permits us in the majority of cases a numerical evaluation of probabilities that has exactly the same characteristics as the evaluation of prices by the method of exchange" (Borel, 1924/1964, p. 57). The betting method provides the operative tool by which one can build a bridge between probability and action, as it "can be applied to all verifiable judgments; it allows for a numerical evaluation of probabilities with a precision quite comparable to that with which one evaluates prices" (Borel, 1924/1964, p. 57).

Borel endorses Reichenbach's adhesion to the pragmatist principle that the meaning of a proposition lies with its practical consequences. In Borel's words: "a proposition has practical interest for men only insofar as it can influence their actions" (Borel, 1939/1952, p. 89). At the same time, he moves a number of objections against the frequency theory. In particular, he objects to Reichenbach's solution to the single case problem that the homogeneity requirement would lead us to consider "classes that contain so few elements that the concept of frequency no longer applies" (Borel, 1939/1952, p. 87). The more detailed the description of a single case is, the more evident the differences with other cases of the same kind, so that "one will find that probability is defined in a way that is less and less precise the better the case at hand is known" (Borel, 1939/1952, p. 88). Albeit agreeing with Reichenbach that single case probability evaluations are of vital importance for the sake of practical applications, Borel deems that frequencies are useful, but other elements also come into play. He makes the example of a doctor asked to predict the probability of survival of a patient that had contracted a certain disease. Surely the doctor will consider the frequency of deaths among people with the same illness in a given period of time, but he will likewise take into account additional information considered relevant in the light of his own experience. Borel's conclusion is that single case probability attributions result from the concurrence of empirical information, especially frequencies, plus personal experience and common sense.

Given that subjective ingredients are part of probability evaluations, the possibility that based on the same body of empirical information two people can come up with different assignments is admitted, precisely as it was by subjectivists. When it comes to measuring the probability of single events, Borel holds that "the probability of a single case is defined subjectively by the conditions under which an agent is ready to bet on the occurrence or non-occurrence of an event" (Borel, 1939/1952, p. 105). It is noteworthy in that connection that he addresses the reader to de Finetti's "La prévision, ses lois logiques, ses sources subjectives".[11]

Having so accounted for the probability of single events Borel feels the need to address the issue of defining an objective notion of probability. Objective probabilities, he claims, "can be defined as probabilities whose value is the same for a certain number of individuals who are well informed on the conditions of the aleatory event" (Borel, 1939/1952, p. 105). It should not pass unnoticed that Borel's concept of objective probability strongly resembles that put forward by Frank Ramsey, who defines the probabilities occurring in physics as being objective in the sense "that everyone agrees about them, as opposed e.g. to odds on horses" (Ramsey, 1990, p. 106). The idea is that the assessment of probability occurring in physics is constrained by theories that have gained the assent of the scientific community after a good deal of evidence in their favour has been collected. Late in his life, de Finetti also admitted that in some areas of science, like physics, probability assignments stand on "more solid grounds" than those belonging to everyday life, but he did not develop that idea (de Finetti, 1995/2008, p. 63). Apart from that remark, de Finetti did not pay much attention to the notion of objective probability, convinced that subjective probability can do the whole job. As a matter of fact, Bruno de Finetti praises Borel for holding that probability must be referred to the single case and can be measured sufficiently well by means of the betting method. At the same time, de Finetti criticizes Borel's eclectic attitude according to which probability can take an objective as well as subjective value (see de Finetti, 1939).

It is noteworthy that after defining objective probability as recalled above Borel adds that "should an event, like the throw of a die, be repeatable a great number of times under the same conditions, the theory of repeated trials tells us that the limiting value of the frequency equals the probability; this will give us a verification, *not a definition*"

[11] See de Finetti (1937/1964).

(italics added, Borel, 1939/1952, p. 105). This claim suggests a parallel with the distinction drawn by de Finetti between the *definition* and the *evaluation* of probability, stating that while probability is by definition the expression of subjective degree of belief, its evaluation is a complex procedure which results from "the conjunction of both objective and subjective elements at our disposal" (de Finetti, 1973, p. 366). Like de Finetti, Borel stresses that what makes the difference among probability evaluations is the amount and kind of evidence backing them, but he refuses to conflate the meaning of probability with the objective elements - be it frequencies and/or symmetries - that are part of that evidence.

To sum up, Borel can be deemed a subjectivist whose conception of probability has much in common with de Finetti's, but his position is more moderate due to his admission that probability assessments made in the context of sciences like physics have an objective character. However, Borel does not attach a realistic meaning to objective probability, taking instead an attitude closer to the pragmatist idea of objectivity as agreement among members of the scientific community.

2.3 Paul Lévy's "rationalistic theory"

Paul Lévy (1886-1971), professor of analysis at the École Polytechnique in Paris, gave important contributions to various branches of mathematics, including functional analysis and probability theory. In *Calcul des probabilités* 1925 Lévy addresses the foundations of probability, putting forward the viewpoint he calls "rationalistic theory" (*théorie rationaliste*), later summarized in a short article entitled "Les fondements du calcul des probabilités", appearing in 1949 in the already mentioned issue of the journal *Dialectica* dealing with "probable knowledge". Like Fréchet and Borel, Lévy is concerned with the *practical meaning* of probability, taking for granted that its mathematical aspects are beyond dispute. Clarifying the relationships between the mathematical principles (axioms) and their practical applications is precisely the task Lévy attaches to the debate on the foundations of probability, which must be of interest to both philosophers and mathematicians.

Lévy takes a sympathetic attitude towards the notion that probability has a subjective component, and rejects von Mises' frequency theory, called *empirical*. In his words: "probability, essentially subjective, traces a clear-cut distinction between what one knows and what one does not know but can be modified by some information that was previously unavailable" (Lévy, 1925, p. 14). The proper tool for such updating is Bayes' rule, to which Lévy devotes one section of the first chapter of *Calcul des probabilités*.

The meaning of probability varies according to the problem addressed: one thing is to talk about the probability of obtaining a double six when two dice are thrown, and another thing is to talk about the probability of contracting a certain disease in a given situation. While the second problem "is of interest to statisticians and doctors [...] mathematicians and philosophers should primarily concern themselves with games of chance, where the concept of probability can be seen in its purity" (Lévy, 1949, p. 57). The meaning of probability in the realm of chance games is "the expectation of a frequency": we expect that after a considerably long series of throws each side of a die will come up with a frequency close to 1⁄6, unless we have reason to believe otherwise, namely assuming that the die is unbiased. What we expect is not that the die comes up *exactly* 1⁄6th of the times it has been thrown, but that this result obtains *approximately*. More precisely, we expect that the deviation between the observed and the expected frequency is not too large. If not, we must revise our expectations. Conversely, if the deviation between the observed frequency and our expectation is small, our initial expectation is confirmed, together with the assumption backing it. Fréchet calls attention to the analogy between this way of proceeding and what is commonly done in geometry where all reasoning is based on perfect solids which are not to be found in reality, but good approximations are considered satisfactory. This inspires Lévy's claim that "only subjective probability is liable to offer a schematization of probability which is as useful to the probabilist as the consideration of perfect solids is to the geometrician. One can discuss the intuitive character of subjective probability [. . .] But I think that everyone must acknowledge the interest of subjective probability" (Lévy, 1949, pp. 58–59).

Against the pretence of frequentists of grounding the evaluation of probability on frequencies alone Lévy calls attention to the importance of intuition and individual judgment, on which rests the task of applying the laws of probability to practice. To the upholders of the frequency theory like von Mises, whom he labels "empiricists", Lévy objects that we only experience individual cases, and when a series of repeated trials is available "if in the result of such experiences we discover a characteristic which does not look random, but [...] predictable, this must be so by virtue of individual experience. It must be discovered. The empiricist denies it. Therefore I can only reckon empiricism as the refusal of a progress that rationalism has made without effort" (Lévy, 1949, p. 62). The fundamental mistake made by those who embrace the frequency theory is that of putting at the core of the definition of probability the relationship between probability and frequency, which is thereof assumed *a priori*, when it should be a matter for demonstration. In Lévy's words: "the empiricist cannot hope to demonstrate Bernoulli's theorem, because he starts with assuming the property that should be proved" (Lévy, 1949, p. 63).

To admit that subjective elements enter into the evaluation of probability is for Lévy simply a matter of taking a rationalistic attitude towards the issue of the interpretation of probability, which boils down to the problem of linking theory and practice. In Lévy's words, "There is no doubt that the rationalist has his own difficulties when it comes to dealing with empirical results. But he has never concealed such difficulties; to solve them he will employ all his good sense, he will not merely use a ready-made formula" (Lévy, 1949, p. 64). The rationalist's decisive advantage over the empiricist is that of having understood that the properties of a series must be explicated in terms of the properties of its elements. Lévy considers it crucial progress, and deems the empiricist's refusal to acknowledge it a "fundamental mistake" (Lévy, 1949, p. 64).

3 Closing remarks

As claimed at the outset, the work of the French mathematicians Fréchet, Borel and Lévy is best seen in the frame of a larger picture that includes a number of authors active in different fields and in different places, such as Frank Ramsey and Harold Jeffreys in Great Britain, Bruno de Finetti in Italy, Richard von Mises and Hans Reichenbach in Germany, and Janina Hosiasson in Poland, all of whom heralded a genuinely probabilistic view of knowledge. The same conviction was actually shared by many others, including authors like John Maynard Keynes and Ernest Nagel, thereby extending the picture considerably.

A first consideration suggested by the work of such authors is that the debate on the foundations of scientific knowledge is broader than the received view centred on the Vienna Circle that has long been dominant.

A further consideration that can be drawn from their writings is that the upholders of probabilistic epistemology share by and large a body of tenets ingrained in the pragmatist tradition, such as the stress on prediction as the main task of science, the idea that success is the canon for the justification of induction, the centrality of action in connection with the theory of meaning and the value of probability, and the attention paid to the practical applications of probability. This is evidence of the influence exercised by pragmatism on European epistemology. Reichenbach, Ramsey, Hosiasson and de Finetti all explicitly acknowledge the influence on their thought of pragmatist thinkers such as Peirce, James and the Italian Giovanni Vailati.[12] By contrast, the French probabilists do not refer to such authors, but their writings abound with references to Henri Poincaré. There is no doubt that Poincaré was a formidable source of inspiration for French mathematicians,[13] and exercised an influence that went far beyond technical aspects, deeply affecting their philosophy of probability whose pragmatist flavour is likely to descend from his work.

[12] See Galavotti (2011b) and Galavotti (to appear) for more on the influence of pragmatism on the debate on the foundations of probability in the last century.

[13] See for instance von Plato (1994), where Borel is regarded as Poincaré's successor not only academically, but also "in an intellectual sense" (p. 36).

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